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POWER TYPE WEIGHTED GRAND VARIABLE EXPONENT SMIRNOV CLASSES AND APPROXIMATION BY RATIONAL FUNCTIONS

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Abstract: Let G be a doubly connected domain in the complex plane $\mathbb{C}$, bounded by regular curves. In this study the approximation properties of the functions by Faber-Laurent rational functions in the $\rho-$ power weighted grand variable exponent Smirnov classes $\mathrm{W}^{p(.),\theta}(G,\omega)$, $\theta>0$ in the terms of the $rth, r=1,2...$ mean modulus of smoothness are investigated.

Keywords: Faber-Laurent rational functions, regular curve, weighted grand variable exponent Smirnov classes, r -th mean modulus of smoothness.

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1. Introduction

Let $\mathbb{T}$ denote the interval $[0,2\pi], \omega:\mathbb{T}\to\mathbb{R}$ be a weight function, i.e., almost everywhere (a.e.) positive and integrable function on $\mathbb{T}$ and $L^P(\mathbb{T}), 1\le p\le\infty$, the Lebesgue space of measurable functions on $\mathbb{T}$.

Let us denote by $\wp$ the class of Lebesgue measurable functions $p:\mathbb{T}\to(1,\infty)$ such that $1<p_*:=ess\inf_{x\in\mathbb{T}} p(x)\le p^*:=ess\sup_{x\in\mathbb{T}} p(x)<\infty.$ The conjugate exponent of $p(x)$is shown by $p'(x):=\dfrac{p(x)}{p(x)-1}.$ For $p(.)\in\wp(\mathbb{T})$, we define a class $L^{p(.)}(\mathbb{T})$ of 2π periodic measurable functions $f:\mathbb{T}\to\mathbb{R}$ satisfying the condition

$$\int_{\mathbb{T}} |f(x)|^{p(x)} dx < \infty .$$

This class $L^{P(\cdot)}(\mathbb{T})$ is a Banach space with respect to the norm

$$\| f \|_{L^{p(\cdot)}(\mathbb{T})} := \inf \left\{ \lambda > 0 : \int_{\mathbb{T}} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\} .$$

We say that a function $p(.) \in \wp(\mathbb{T})$ belongs to the class $\wp^{\log}(\mathbb{T})$ if there is a positive constant C such that for all $x, y \in \mathbb{T}$ with $|x-y| \leq 1/2$

$$|p(x) - p(y)| \leq \frac{C}{-\ln|x-y|} .$$

The spaces $L^{P(\cdot)}(\mathbb{T})$ are called *generalized Lebesgue spaces with variable exponent*. It is know that for $p(x) := p(0 < p \leq \infty)$, the space $L^{P(\cdot)}(\mathbb{T})$ coincides with the Lebesgue space $L^P(\mathbb{T})$. If $p^* < \infty$ then the spaces $L^{P(\cdot)}(\mathbb{T})$ represent a special case of the so-called Orlicz-Musielak spaces [45]. For the first time Lebesgue spaces with variable exponent was introduced by Orlicz [46]. Note that the generalized Lebesgue spaces with variable exponents are used in the theory of elasticity, in mechanics, especially in fluid dynamics for the modelling of electrorheological fluids, in the theory of differential operators, and in variational calculus [10], [11], [12], [48] and [50]. Detailed information about properties of the Lebesgue spaces with variable exponent can be found in [6], [14], [41], [44], [49] and [56]. Note that, some of the fundamental problems of the approximation theory in the generalized Lebesgue spaces with variable exponent of periodic and non-periodic functions were studied and solved by Sharapudinov [55], [56] and [57].

A function $\omega : \mathbb{T} \to [0, \infty]$ is called a *weight function* if ω is a measurable and almost everywhere (a. e.) positive.

Let ω be a 2π periodic weight function. We denote by $L^P(\mathbb{T}, \omega), 1 < p < \infty$ the *weighted Lebesgue space* of 2π periodic measurable functions $f : \mathbb{T} \to \mathbb{C}$ such that $f \omega^{\frac{1}{p}} \in L^p(\mathbb{T})$. For f $f\omega \in L^p(\mathbb{T}, \omega)$ we set

$$\| f \|_{L^p(\mathbb{T},\omega)} := \left\| f \omega^{\frac{1}{p}} \right\|_{L^p(\mathbb{T})} .$$

Let $\Gamma \subset \mathbb{C}$ be a Jordan rectifiable curve and let $\omega : \Gamma \to [0,\infty]$ be a weight function, that is a positive almost everywhere (a.e.) and integrable function on Γ. For $1 < p < \infty$ we define a class $L^p(\Gamma,\omega)$ of Lebesgue measurable functions f on Γ satisfying the condition

$$\left(\frac{1}{|\Gamma|} \int_{\Gamma} \left| f(z) \right|^p \omega(z) |dz| < \infty \right)^{\frac{1}{p}} < \infty ,$$

where $|\Gamma|$ is the length of Γ. We denote by $L^{p),\theta}(\Gamma,\omega)$, $\theta > 0, 1 < p < \infty$ the Lebesgue space of all measurable functions f on Γ, that is, the space of all such functions for which

$$\| f \|_{L^{p),\theta}(\Gamma,\,\omega)} := \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon^{\theta}}{|\Gamma|} \int_{\Gamma} \left| f(z) \right|^{p-\varepsilon} \omega(z) |dz| \right)^{\frac{1}{p-\varepsilon}} < \infty .$$

The space $L^{p),\theta}(\Gamma,\omega)$ is called the *weighted generalized grand Lebesgue spaces*. $L^{p),\theta}(\Gamma,\omega)$ is the Banach function space, non-reflexive, non-separable and non-rearrangement. The grand and generalized grand Lebesgue space were introduced in the works [33] and [23], respectively. If $\theta_1 < \theta_2$ then for $0 < \varepsilon < p-1$ the embeddings

$$L^p(\Gamma,\omega) \subset L^{p),\theta_1}(\Gamma,\omega) \subset L^{p),\theta_2}(\Gamma,\omega) \subset L^{p-\varepsilon}(\Gamma,\omega), 1 < p < \infty$$

hold. Note that the information about properties and applications of the grand Lebesgue spaces can be found in [7], [8], [20], [23], [33], [40], [52] and [53].

Let $\theta > 0$ and $p(.) \in \wp(\mathbb{T})$.We denote by $L^{p(.),\theta}(\mathbb{T})$ the *grand variable exponent Lebesgue space* of all measurable functions f on $\mathbb{T}$, that is, the space of all such functions for which

$$\| f \|_{L^{p(.),\theta}(\mathbb{T})} := \sup_{0<\varepsilon<p_*-1} \varepsilon^{\frac{\theta}{p_*-\varepsilon}} \| f \|_{L^{p(.)-\varepsilon}(\mathbb{T})} < \infty$$

Note that when $p(.) = p$ is a constant function, these spaces coincide with the grand Lebesgue spaces $L^{p),\theta}(\mathbb{T})$.

If $0 < \varepsilon < p_* - 1$, it is easy to see that the following continuous embeddings
hold

$$L^{p(\cdot)}(\mathbb{T}) \subset L^{p(\cdot),\theta}(\mathbb{T}) \subset L^{p(\cdot)-\varepsilon}(\mathbb{T}) \subset L^{1}$$

due to $|\mathbb{T}| < 1$ (see [19], [42])

Let ω be a weight function. on $\mathbb{T}$ and $p(.) \in \rho(\mathbb{T})$. The *weighted grand variable exponent Lebesgue spaces* $L^{p(\cdot),\theta}(\mathbb{T},\omega)$ is the class of all measurable functions f for which

$$\| f \|_{L^{p(\cdot),\theta}(\mathbb{T},\omega)} := \sup_{0<\varepsilon<p_*-1} \varepsilon^{\frac{\theta}{p_*-\varepsilon}} \| f \|_{L^{p(\cdot)-\varepsilon}(\mathbb{T},\omega)} < \infty$$

Note that for $f \in L^{p(\cdot)}(\mathbb{T},\omega)$ the norm equality $\| f \|_{L^{p(\cdot),\theta}(\mathbb{T},\omega)} = \left\| f\,\omega^{\frac{1}{p(\cdot)}} \right\|_{L^{p(\cdot),\theta}(\mathbb{T})}$ is not valid in $L^{p(\cdot),\theta}(\mathbb{T},\omega)$ (see [21]).

If $0 < C \le \omega,\ p(.) \in \wp(\mathbb{T})$ and $\theta_1 < \theta_2$ then for $0 < \varepsilon < p_* - 1$ the following continuous embeddings hold

$$L^{p(\cdot)}(\mathbb{T},\omega) \subset L^{p(\cdot),\theta_1}(\mathbb{T},\omega) \subset L^{p(\cdot),\theta_2}(\mathbb{T},\omega) \subset L^{p(\cdot)-\varepsilon}(\mathbb{T},\omega) \subset L^{p(\cdot)-\varepsilon}(\mathbb{T}) \subset L^{1}$$

due to (see [19], [42]).

Let h be a continuous function and let

$$\omega(h,t) := \sup_{\|t_1-t_2\|<t} |h(t_1) - h(t_2)|,\ t > 0$$

be its modulus of continuity.

Let Γ be a smooth Jordan curve and let $\theta(s)$ be the angle between the tangent and the positive real axis expressed as a function of arc length *s*. If Γ has a modulus of continuity $\omega(\theta, s)$ satisfied the Dini smooth condition

$$\int_0^{\delta} \frac{\omega(\theta,s)}{s}\, ds < \infty,\ \ \delta > 0\,,$$

then we say that Γ is a *Dini smooth* curve.

A Jordan curve Γ is called *regular,* if there exists a number $c > 0$ such that for every $r > 0$, $\sup\{|\Gamma \cap D(z,r)| : z \in \Gamma\} \le cr$, where $D(z,r)$ is an open disk with radius r and centered at z and $\left|\Gamma\cap D(z,r)\right|$ is the length of the set $\Gamma\cap D(z,r)$.

Let $p(.)\in\wp(\mathbb{T})$ and let ω be a weight function on $\mathbb{T}$. ω is said to satisfy Muckenhoupt's $A_{p(.)}-$condition on $\mathbb{T}$ if

$$\sup_{B_j}\left|B_j\right|^{-1}\left\|\omega\chi_{B_j}\right\|_{p(.)}\left\|\omega^{-1}\chi_{B_j}\right\|_{p'(.)}<\infty,\quad 1/p(.)+1/p'(.)=1$$

where supremum is taken over all open intervals $B_j\subset\mathbb{T}$ with the characteristic functions χ_{B_j}.

Let us further assume that B is a simply connected domain with a rectifiable Jordan boundary Γ and $B^-:=ext\Gamma$. Without loss of generality we assume that $0\in B$. Let

$$\mathbb{T}=\{\omega\in\mathbb{C}:|\omega=1|\},\ D:=int\mathbb{T},\ D^-:=ext\mathbb{T}$$

Also, ϕ^* stand for the conformal mapping of B^- onto D^- normalized by

$$\phi^*(\infty)=\infty$$

and

$$\lim_{z\to\infty}\frac{\phi^*(z)}{z}>0,$$

and let ψ^* be the inverse of ϕ^*. Let ϕ_1^* be the conformai mapping of B onto D^-, normalized by

$$\phi_1^*(0)=\infty$$

and

$$\lim_{z\to0}z\phi_1^*(z)>0.$$

The inverse mapping of ϕ_1^* will be denoted by ψ_1^*.

Note that the mappings ψ^* and ψ^* have in some deleted neighborhood of ∞ representations

$$\psi^*(\omega)=\alpha\omega+\alpha_0+\frac{\alpha_1}{\omega}+\frac{\alpha_2}{\omega^2}+...+\frac{\alpha_k}{\omega^k}+...,\ \alpha>0$$

and

$$\psi_1^*(\omega)=\frac{\beta_1}{\omega}+\frac{\beta_2}{\omega^2}+...+\frac{\beta_k}{\omega^k}+..., \beta_1>0.$$

We set $p_0^*=p\left(\psi^*(\omega)\right)$ and $p_1^*=p\left(\psi_1^*(\omega)\right)$. For $0<\varepsilon<p(.)-1$ the functions

$$\frac{\left(\frac{d\psi^*(\omega)}{d\omega}\right)^{1-\frac{1}{p_0^*-\varepsilon}}}{\psi^*(\omega)-z},\ z\in B,$$

and

$$\frac{\omega^{-\frac{2}{p_1^*-\varepsilon}}\left(\frac{d\psi_1^*(\omega)}{d\omega}\right)^{1-\frac{1}{p_1^*-\varepsilon}}}{\psi_1^*(\omega)-z},\ z\in B^-.$$

are analytic in the domain D.

Let $p(.)\in\wp^{\log}\left(B^-\right)$. The following expansions hold :

$$\frac{\left(\frac{d\psi^*(\omega)}{d\omega}\right)^{1-\frac{1}{p_0^*-\varepsilon}}}{\psi^*(\omega)-z}=\sum_{k=0}^{\infty}\frac{\Phi_{k,p(.)-\varepsilon}(z)}{\omega^{k+1}},\ z\in B, \omega\in D^-$$

and

$$\frac{\omega^{-\frac{2}{p_1^*-\varepsilon}}\left(\frac{d\psi_1^*(\omega)}{d\omega}\right)^{1-\frac{1}{p_1^*-\varepsilon}}}{\psi_1^*(\omega)-z}=\sum_{k=1}^{\infty}-\frac{F_{k,p(.)-\varepsilon}\left(\frac{1}{z}\right)}{\omega^{k+1}},\ z\in B^-, \omega\in D^-,$$

where $\Phi_{k,p(.)-\varepsilon}(z)$ and $F_{k,p(.)-\varepsilon}\left(\frac{1}{z}\right)$ are the $p(.)-\varepsilon$ Faber polynomials of degree k with respect to z and $\frac{1}{z}$ for the continuums $\overline{B}$ and $\overline{B^-}$, respectively (see also [9], [29], [31] and [51, pp. 255-257]).

Let $E^1(B)$ be a classical Smirnov class of analytic functions in B. The set $E^{P),\theta}(B,\omega):=\left\{f\in E^1(B): f\in L^{p),\theta}(\Gamma,\omega)\right\}$ is called the $\omega-$*weighted* *generalized grand Smirnov class in* B.

We denote the closure of $L^{p(.)}(\mathbb{T},\omega)$ by $L^{p(.),\theta}(\mathbb{T},\omega)$. If $f\in L^{p(.),\theta}(\mathbb{T},\omega)$ then according to [19] we have

$$\lim_{\varepsilon\to 0}\varepsilon^{\frac{\theta}{p_*-\varepsilon}}\|f\|_{L^{p(.)-\varepsilon}(\mathbb{T},\omega)}=0$$

We denote the Hardy-Littlewood maximal operator Mf of f by

$$M\ f(x)=\sup_I \frac{1}{|I|}\int_I |f(x)|dx,\ x\in\mathbb{T},$$

where the supremum is taken over all intervals I whose length is less than 2π.

We consider power weights of the form $\rho(t)=|t-t_0|^{\gamma}$, $t_0\in\mathbb{T}$, $-1<\gamma<(p(t_0)-1)$. Note that if $p(.)\in\wp(\mathbb{T}), t_0\in(-\pi,\pi), \rho(t)=|t-t_0|^{\gamma}$ and $-1<\gamma<(p(t_0)-1)$, then according to [18] the Hardy-Littlewood maximal operator Mf is bounded in $L^{p(.),\theta}(\mathbb{T},\rho), \theta>0$.

The function $\rho(t)=|t-t_0|^{\gamma}, -1<\gamma<(p(t_0)-1)$ satisfies the $A_{p(.)}$ condition of Muckenhoupt weights. Therefore, according to [4] we have the continuous embedding $L^{p(.),\theta}(\mathbb{T},\rho)\subset L^1(\mathbb{T})$.

Let $p(.)\in\wp(\mathbb{T}), \rho(t)=|t-t_0|^{\gamma}, -1<\gamma<(p(t_0)-1)$ and $\theta>0$. For $f\in L^{p(.),\theta}(\mathbb{T},\rho)$ we define the operator

$$(v_h^r f)(x):=\frac{1}{h}\int_0^h|\Delta_t^r f(x)|dt,\ h>0$$

where

$$\Delta_t^r f(x):=\sum_{k=0}^{r}(-1)^{r+k+1}\binom{r}{k} f(\omega e^{ikt}), r\in\mathbb{N}=\{\theta 1,2,...\},\ t>0,$$

Note that the operator v_h is a bounded on $L^{p(.),\theta}(\mathbb{T},\rho)$ [5]:

$$\sup_{|h|\le\delta}\|v_h^r(f)\|_{L^{p(.),\theta}(\mathbb{T},\rho)}\le c_1\|f\|_{L^{p(.),\theta}(\mathbb{T},\rho)},\ \delta>0.$$

Let $f\in L^{p(.),\theta}(\mathbb{T},\rho), \theta>0$ and $p(.)\in\wp(\mathbb{T})$. The function

$$\Omega^r_{p(.),\theta,\rho}(f,\delta) := \sup_{|h|\le\delta} \left\| \nu^r_h f(\omega) \right\|_{L^{p(.),\theta}(\mathbb{T},\rho)}, \ \delta > 0$$

is called *the r - th mean modulus* of $f \in L^{p(.),\theta}(\mathbb{T},\rho)$.

It can be easily shown that $\Omega^r_{p(.),\theta,\rho}(f,\cdot)$ is a continuous, nonnegative and nondecreasing function satisfying the conditions [5]

$$\lim_{\delta\to 0}\Omega^r_{p(.),\theta,\rho}(f,\delta)=0,\ \Omega^r_{p(.),\theta,\rho}(f+g,\delta)\le\Omega^r_{p(.),\theta,\rho}(f,\delta)+\Omega^r_{p(.),\theta,\rho}(g,\delta),\ \delta>0$$

for $f, g \in L^{p(.),\theta}(\mathbb{T},\rho)$.

Let G be a doubly connected domain in the complex plane $\mathbb{C}$, bounded by the rectifiable Jordan curves Γ_1 and Γ_2 (the closed curve Γ_2 is in the closed curve Γ_1). Without loss of generality we assume $0 \in \operatorname{int}\Gamma_2$. Let $G^0_1 := \operatorname{int}\Gamma_1$, $G^\infty_1 := \operatorname{ext}\Gamma_1$, $G^2_0 := \operatorname{int}\Gamma_2$, $G^\infty_2 := \operatorname{ext}\Gamma_2$.

We denote by $\omega = \phi(z)$ the conformal mapping of G^∞_1 onto domain D^- normalized by the conditions

$$\phi(\infty)=\infty,\ \lim_{z\to\infty}\frac{\phi(z)}{z}=1$$

and let ψ be the inverse mapping of ϕ.

We denote by $\omega = \phi_1(z)$ the conformal mapping of G^0_2 onto domain D^- normalized by the conditions

$$\phi_1(0)=\infty,\ \lim_{z\to 0}\left(z.\phi_1(z)\right)=1,$$

and let ψ_1 be the inverse mapping of ϕ_1.

Let us take

$$C_{\rho_0} := \left\{ z : \left|\phi(z)\right| = \rho_0 > 1 \right\},\ \Gamma_{r_0} := \left\{ z : \left|\phi_1(z)\right| = r_0 > 1 \right\}.$$

For $\Phi_{k,p(.)-\varepsilon}(z)$ and $\Phi_{k,p(.)-\varepsilon}\left(\frac{1}{z}\right)$ the following integral representations hold [9], [29], [31] and [51, pp.255-257]:

1) If $z \in \operatorname{int} C_{\rho_0}$, then

$$\Phi_{k,p(.)-\varepsilon}(z) = \frac{1}{2\pi i}\int_{C_{\rho_0}} \frac{\left[\phi(\zeta)\right]^k \left(\phi'(\zeta)\right)^{\frac{1}{p(\zeta)-\varepsilon}}}{\zeta - z} d\zeta \qquad (1)$$

2) If $z \in extC_{\rho_0}$, then

$$\Phi_{k,p(.)-\varepsilon}(z) = \left[\phi(z)\right]^k \left(\phi'(z)\right)^{\frac{1}{p(z)-\varepsilon}} + \frac{1}{2\pi i}\int_{C_{\rho_0}} \frac{\left[\phi(\zeta)\right]^k \left(\phi'(\zeta)\right)^{\frac{1}{p(\zeta)-\varepsilon}}}{\zeta - z} d\zeta \tag{2}$$

3) If $z \in \operatorname{int} C_{r_0}$, then

$$F_{k,p(.)-\varepsilon}\left(\frac{1}{z}\right) = \left[\phi_1(z)\right]^{k-\frac{2}{p(z)-\varepsilon}} \left(\phi_1'(z)\right)^{\frac{1}{p(z)-\varepsilon}} - \frac{1}{2\pi i}\int_{C_{r_0}} \frac{\left[\phi_1(\zeta)\right]^{k-\frac{2}{p(\zeta)-\varepsilon}} \left(\phi_1'(\zeta)\right)^{\frac{1}{p(\zeta)-\varepsilon}}}{\zeta - z} d\zeta \tag{3}$$

4) If $z \in extC_{r_0}$, then

$$F_{k,p(.)-\varepsilon}\left(\frac{1}{z}\right) = -\frac{1}{2\pi i}\int_{C_{r_0}} \frac{\left[\phi_1(\zeta)\right]^{k-\frac{2}{p(\zeta)-\varepsilon}} \left(\phi_1'(\zeta)\right)^{\frac{1}{p(\zeta)-\varepsilon}}}{\zeta - z} d\zeta \,. \tag{4}$$

We set $p_0 = p(\psi(\omega))$ and $p_1 = p(\psi_1(\omega))$. If a function $f(z)$ is analytic in the doubly connected domain bounded by the curves C_{p_0} and Γ_{r_0}, then the fallowing series expansion holds:

$$f(z) = \sum_{k=0}^{\infty} a_k \Phi_{k,p(.)-\varepsilon}(z) + \sum_{k=1}^{\infty} b_k F_{k,p(.)-\varepsilon}\left(\frac{1}{z}\right), \tag{5}$$

where

$$a_k = \frac{1}{2\pi i}\int_{|\omega|=\rho_1} \frac{f\left[\psi(\omega)\right]\left(\psi'(\omega)\right)^{\frac{1}{p_0-\varepsilon}}}{\omega^{k+1}} d\omega, \ \left(1 < \rho_1 < \rho_0\right), \ k = 0,1,2...$$

and

$$b_k = \frac{1}{2\pi i}\int_{|\omega|=r_1} \frac{f\left[\psi_1(\omega)\right]\left(\psi_1'(\omega)\right)^{\frac{1}{p_1-\varepsilon}} \omega^{\frac{2}{p_1-\varepsilon}}}{\omega^{k+1}} d\omega, \left(1 < r_1 < r_0\right), k = 1,2...$$

The series (5) is called the $p(.)-\varepsilon$ Faber -Laurent series of f, and the coefficients a_k and b_k are said to be the $p(.)-\varepsilon$ Faber -Laurent coefficients of f. For $z \in G$ by Cauchy's integral formulae we have

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i}\int_{\Gamma_2} \frac{f(\xi)}{\xi - z} d\xi \,.$$

If $z \in \operatorname{int}\Gamma_2$ and $z \in ext\Gamma_1$, then

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i}\int_{\Gamma_2}\frac{f(\xi)}{\xi - z}d\xi = 0. \qquad (6)$$

Let us consider

$$I_1(z) := \frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(\zeta)}{\zeta - z}d\zeta,\ I_2(z) := \frac{1}{2\pi i}\int_{\Gamma_2}\frac{f(\xi)}{\xi - z}d\xi.$$

The function $I_1(z)$ determines the functions $I_1^+(z)$ and $I_1^-(z)$ while the function $I_2(z)$ determines the functions $I_2^+(z)$ and $I_2^-(z)$. The functions $I_1^+(z)$ and $I_1^-(z)$ are analytic in $\operatorname{int}\Gamma_1$ and $ext\Gamma_1$, respectively. The functions $I_2^+(z)$ and $I_2^-(z)$ are analytic in $\operatorname{int}\Gamma_2$ and $ext\Gamma_2$, respectively.

Let B be a finite domain in the complex plane bounded by a rectifiable Jordan curve Γ and $f \in L_1(\Gamma)$. Then the functions f^+ and f^- defined by

$$f^+(z) := \frac{1}{2\pi i}\int_{\Gamma}\frac{f(\zeta)}{\zeta - z}d\zeta,\ z \in B$$

and

$$f^-(z) := \frac{1}{2\pi i}\int_{\Gamma}\frac{f(\zeta)}{\zeta - z}d\zeta,\ z \in B^-$$

are analytic in B and B^- respectively, and $f^-(\infty) = 0$. Thus the limit

$$S_\Gamma(f)(z) := \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\Gamma\cap\{\zeta:|\zeta - z|>\varepsilon\}}\frac{f(\zeta)}{\zeta - z}d\zeta$$

exists and is finite for almost all $z \in \Gamma$.

The quantity $S_\Gamma(f)(z)$ is called the Cauchy singular integral of f at $z \in \Gamma$.

According to the Privalov theorem [22, pp.431], if one of the functions f^+ or f^- has the non-tangential limits a.e. on Γ, then $S_\Gamma(f)(z)$ exists a.e. on Γ and also the other one has the non-tangential limits a.e. on Γ. Conversely, if

$S_\Gamma(f)(z)$ exists a.e. on Γ, then the functions $f^+(z)$ and $f^-(z)$ have non-tangential limits a.e. on Γ. In both cases, the formulae

$$f^+(z) = S_\Gamma(f)(z)+\frac{1}{2}f(z), \quad f^-(z) = S_\Gamma(f)(z)-\frac{1}{2}f(z)$$

and hence

$$f = f^+ - f^- \tag{7}$$

holds a.e. on Γ.

Let Γ be a regular Jordan curve. Assume that $p(.)\in \wp^{\log}(\Gamma)$ and $\rho(z)=|z-z_0|^\gamma$, $z_0\in\Gamma$, $\frac{1}{p(z_0)}<\gamma<\frac{1}{p'(z_0)}$. Then from the results given in [43], it follows that singular integral $S_\Gamma(f)$ is bounded on $L^{p(.),\theta}(\Gamma,\rho)$, $\theta>0$.

We will say that the doubly connected domain G is bounded by the regular curve if the domains G_1^0 and G_2^0 are bounded by the closed regular curves.

Let $\Gamma_i\ (i=1,2)$ be a regular curve. We set $p_0(\omega):=p(\psi(\omega))$, $p_1(\omega):=p(\psi_1(\omega))$,

$$f_0 := f[\psi_1(\omega)](\psi'(\omega))^{\frac{1}{p_0-\varepsilon}} \text{ for } f\in L^{p(\cdot),\theta}(\Gamma_1,\omega) \text{ and let}$$

$$f_1 := f[\psi_1(\omega)](\psi_1'(\omega))^{\frac{1}{p_1-\varepsilon}}\,\omega^{\frac{2}{p_1-\varepsilon}} \text{ for } f\in L^{p(\cdot),\theta}(\Gamma_1,\omega).$$

We also set $\rho_0(\omega):=\rho[\psi(\omega)]$, $\rho_1(\omega):=\rho[\psi_1(\omega)]$. Then, if $f\in L^{p(\cdot),\theta}(\Gamma_1,\rho)$ and $f\in L^{p(\cdot),\theta}(\Gamma_2,\rho)$ we obtain $f_0\in L^{p_0(\cdot),\theta}(\mathbb{T},\rho_0)$ and $f_1\in L^{p_1(\cdot),\theta}(\mathbb{T},\rho_1)$.

Moreover, $f_0^-(\infty)=f_1^-(\infty)=0$ and by (1.7)

$$\left.\begin{aligned} f_0(\omega) &= f_0^-(\omega)-f_0^-(\omega) \\ f_1(\omega) &= f_1^-(\omega)-f_1^-(\omega) \end{aligned}\right\} \tag{8}$$

a.e. on $\mathbb{T}$.

Now, in the doubly connected domain we define the $\omega-$weighted grand variable exponent Smirnov class. Let $E^1(G)$ be a classical Smirnov class of analytic functions in G. The set $E^{p(\cdot),\theta}(G,\omega):=\left(f\in E^1(G):\, f\in L^{p(\cdot),\theta}(\Gamma,\omega)\right)$ is called the $\omega-$weighted grand variable exponent Smirnov class in G. We denote by $\mathrm{W}^{p(.),\theta}(G,\omega)$ the closure of Smirnov class $E^{p}(G,\omega)$ in the space $E^{p(.),\theta}(G,\omega)$.

Using the proof scheme developed in the work [30, Lemma 3] we can prove the following Lemma.

Lemma 1.1. *Let*

$$\rho(t)=|t-t_0|^{\gamma}, -1<\gamma<(p(t_0)-1),\ t_0\in\mathbb{T},\ p(.)\in\wp^{\log}(\mathbb{T}) \text{ and}$$

$g\in L^{p(.),\theta}(T,\rho),\ \theta>0$. *Then the inequality*

$$\Omega^{r}_{p(.),\theta,\rho}(g^{+},\cdot)\le C\Omega^{r}_{p(.),\theta,\rho}(g,\cdot)$$

holds.

The following Theorem is the disk version of theorem proved in [5].

Theorem 1.2.

Let $\rho(t)=|t-t_0|^{\gamma}, -1<\gamma<(p(t_0)-1),\ t_0\in\mathbb{T},\ p(.)\in\wp(\mathbb{T})$ and $g\in\mathrm{W}^{p(\cdot),\theta}(D,\rho),\ \theta>0$. *If* $\sum_{k=0}^{n}dk(g)\omega^{k}$ *is the nth partial sum of the Taylor series of* g *at the origin, then there exists a constant* $c_2>0$ *such that*

$$\left\|g(\omega)-\sum_{k=0}^{n}d_k(\omega)\omega^{k}\right\|_{L^{p(.),\theta}(\mathbb{T},\rho)}\le c_2\Omega^{r}_{p(.),\theta,\rho}\left(g,\frac{1}{n}\right).\ r\in\mathbb{N}$$

for every natural number n.

We set

$$R_n(f,z):=\sum_{k=0}^{n}a_k\Phi_{k,p(.)-\varepsilon}(z)+\sum_{k=1}^{n}b_kF_{k,p(.)-\varepsilon}\left(\frac{1}{z}\right).$$

The rational function $R_n(f,z)$ is called the $p-\varepsilon$ *Faber-Laurent rational function of degree* n *of* f.

Since series of Faber polynomials are a generalization of Taylor series to the case of a simply connected domain, it is natural to consider the construction of a similar generalization of Laurent series to the case of a doubly-connected domain.

In this study, when the power weight function is of the form $\rho(t)=|z-z_0|^{\gamma}$, $\frac{1}{p(z_0)}<\gamma<\frac{1}{p'(z_0)},\ z_0\in\Gamma$ we study the approximation properties of the functions by Faber-Laurent rational functions in the $p-$ power weighted grand variable exponent Smirnov classes $\mathrm{W}^{p(.),\theta}(G,\omega)$,

$\theta > 0$, defined in the doubly connected domains with the regular boundaries. Similar problems in the different spaces were investigated by several authors (see for example, [1]-[4], [9], [15-19], [24-32], [34-39], [47], [51], [58] and [59]).

We write $a \preceq b$ if $a \le cb$, and $a \asymp b$ if $a \preceq b$ and $b \preceq a$ at the same time.If $a \asymp b$ then we will say that a and b are equivalent.

Our main result can be formulated as following.

Theorem 1.3. *Let G be a finite doubly connected domain with the regular boundary $\Gamma = \Gamma_1 \cup \Gamma_2$, $p(.) \in \wp^{\log}(\Gamma)$, $p_0(.) \in \wp^{\log}(\mathbb{T})$* and *$p_1(.) \in \wp^{\log}(\mathbb{T})$. Assume that $l < \infty$ and $\theta > 0$. Let z_0 be a fixed point on Γ. If*

$\rho(z) = |z - z_0|^{\gamma}$, $\dfrac{1}{p(z_0)} < \gamma < \dfrac{1}{p'(z_0)}$ and $f \in W^{p(.),\theta}(G, \omega)$, then there is a constant $c_3 > 0$ such that for any $n = 1, 2, 3, \ldots$

$$\left\| f - R_n(., f) \right\|_{L^{p(\cdot),\theta}(\Gamma,\rho)} \le c_3 \left\{ \Omega^r_{p_0,\theta,\rho_0}(f_0, 1/n) + \Omega^r_{p_1,\theta,\rho_1}(f_1, 1/n) \right\},$$

where l is a diametr of Γ and *$R_n(., f)$ is the $p(.) - \varepsilon$ Faber-Laurent rational function of degree n of f.*

Note that if the curve Γ is a Dini smooth curve, then since $p(\cdot) \in \wp^{\log}(\mathbb{T})$, $p_0(\cdot) \in \wp^{\log}(\mathbb{T})$ and $p_1(.) \in \wp^{\log}(\mathbb{T})$ are equivalet, the conditions $p_0(\cdot) \in \wp^{\log}(\mathbb{T})$ and $p_1(\cdot) \in \wp^{\log}(\mathbb{T})$ can be removed in Theorem 1.3. In this case the following corollary can be expressed.

Corollary 1.1. *Let G be a finite doubly connected domain with the Dini smooth boundary $\Gamma = \Gamma_1 \cup \Gamma_2$, $p(.) \in \wp^{\log}(\Gamma)$. Assume that $l < \infty$ and $\theta > 0$. Let z_0 be a fixed point on Γ. If $\rho(z) = |z - z_0|^{\gamma}$, $\dfrac{1}{p(z_0)} < \gamma < \dfrac{1}{p'(z_0)}$ and $f \in W^{p(.),\theta}(G, \rho)$, then there is a constant $c_4 > 0$ such that for any $n = 1, 2, 3, \ldots$*

$$\left\| f - R_n(\cdot, f) \right\|_{L^{p(\cdot),\theta}(\Gamma,\rho)} \le c_4 \left\{ \Omega^r_{p_0,\theta,\rho_0}(f_0, 1/n) + \Omega^r_{p_1,\theta,\rho_1}(f_1, 1/n) \right\},$$

where l is a diametr of Γ and *$R_n(., f)$ is the $p(.) - \varepsilon$ Faber-Laurent rational function of degree n of f.*

Note that curve Γ is a Dini smooth curve, similar results were obtained in [30] and [32] studies in Lebesgue spaces variable exponent.

2. Proof of Main Result

Proof of Theorem 1.3. We take the curves Γ_1, Γ_2 and $\mathbb{T} := \{\omega \in \mathbb{C} : |\omega| = 1\}$ as the curves of integration in the formulas (1.2) - (1.5) and (1.6), respectively. (This is possible due to the conditions of theorem 1.1). Let $f \in \varepsilon^{p(.),\theta}(G,\rho)$. Then $f_0 \in \varepsilon^{p_0,\theta}(\mathbb{T},\rho_0)$, $f_1 \in \varepsilon^{p_1,\theta}(\mathbb{T},\rho_1)$. According to (1.8)

$$\left.\begin{aligned} f(\zeta) &= \left[f_0^+(\phi(\zeta)) - f_0^-(\phi(\zeta))\right](\phi(\zeta))^{\frac{1}{p(\zeta)-\varepsilon}} \\ f(\xi) &= \left[f_1^+(\phi_1(\xi)) - f_1^-(\phi_1(\xi))\right](\phi_1(\xi))^{-\frac{2}{p(\xi)-\varepsilon}}(\phi_1'(\xi))^{\frac{1}{p(\xi)-\varepsilon}} \end{aligned}\right\} \qquad (9)$$

Let $z \in ext\Gamma_1$. From (2) and (9) we have

$$\begin{aligned} &\sum_{k=0}^{n} a_k \Phi_{k,p(z)} \\ &= \sum_{k=0}^{n} a_k [\phi(z)]^k (\phi'(z))^{\frac{1}{p(z)-\varepsilon}} + \frac{1}{2\pi i}\int_{\Gamma_1} \frac{(\phi'(\zeta))^{\frac{1}{p(\zeta)-\varepsilon}} \sum_{k=0}^{n} a_k [\phi(\zeta)]^k}{\zeta - z} d\zeta \\ &\qquad = \sum_{k=0}^{n} a_k [\phi(z)]^k (\phi'(z))^{\frac{1}{p(z)-\varepsilon}} \\ &\qquad + \frac{1}{2\pi i}\int_{\Gamma_1} \frac{(\phi'(\zeta))^{\frac{1}{p(\zeta)-\varepsilon}} \sum_{k=0}^{n} a_k [\phi(\zeta)]^k - f_0^+[\phi(\zeta)]}{\zeta - z} d\zeta \\ &\qquad + \frac{1}{2\pi i}\int_{\Gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta - f_0^-[\phi(z)](\phi'(z))^{\frac{1}{p(z)}}. \end{aligned} \qquad (10)$$

For $z \in ext\Gamma_2$, the relations(4) and (9) imply that

$$\sum_{k=1}^{n} b_k F_k\left(\frac{1}{z}\right) = -\frac{1}{2\pi i}\int_{\Gamma_2} \frac{\left(\phi_1'(\xi)\right)^{\frac{1}{p(\xi)-\varepsilon}} \phi_1(\xi)^{-\frac{2}{p(\xi)-\varepsilon}} \sum_{k=1}^{n} b_k \left[\phi_1(\xi)\right]^k}{\xi - z} d\xi$$

$$-\frac{1}{2\pi i}\int_{\Gamma_2} \frac{\sum_{k=0}^{n} b_k \left[\phi_1(\xi)\right]^k}{\xi - z} d\xi \tag{11}$$

$$= \frac{1}{2\pi i}\int_{\Gamma_1} \frac{\left(\phi_1(\xi)\right)^{-\frac{2}{p(\xi)-\varepsilon}} \left(\phi_1'(\xi)\right)^{\frac{1}{p(\xi)-\varepsilon}} \left[f_1^+\left[\phi_1(\xi)\right] - \sum_{k=0}^{n} b_k \left[\phi_1(\xi)\right]^k \right]}{\xi - z} d\xi$$

$$-\frac{1}{2\pi i}\int_{\Gamma_2} \frac{f(\xi)}{\xi - z} d\xi.$$

For $z \in ext\Gamma_1$, taking into account (10), (11) we obtain

$$\sum_{k=0}^{n} a_k \left[\Phi_k(z)\right]^k + \sum_{k=1}^{n} b_k F_k\left(\frac{1}{z}\right)$$

$$= \sum_{k=0}^{n} a_k \left[\phi_k(z)\right]^k \left(\phi'(z)\right)^{\frac{1}{p(z)-\varepsilon}} + \frac{1}{2\pi i}\int_{\Gamma_1} \frac{\left(\phi'(\zeta)\right)^{\frac{1}{p(\zeta)-\varepsilon}} \sum_{k=0}^{n} a_k \left[\phi(\zeta)\right]^k - f_0^+\left[\phi(\zeta)\right]}{\zeta - z} d\zeta$$

$$- f_0^-\left[\phi(z)\right] + \frac{1}{2\pi i}\int_{\Gamma_2} \frac{\left(\phi_1(\xi)\right)^{-\frac{2}{p(\xi)-\varepsilon}} \left(\phi_1'(\xi)\right)^{\frac{1}{p(\xi)-\varepsilon}} \left[f_1^+\left(\phi_1(\xi)\right) - \sum_{k=0}^{n} b_k \left[\phi_1(\xi)\right]^k \right]}{\xi - z} d\xi.$$

Taking limit as $z \to z^* \in \Gamma_1$ along all non-tangential paths outside Γ_1, it appears that

$$f\left(z^*\right)-\sum_{k=0}^{n}a_k\Phi_k\left(z^*\right)-\sum_{k=1}^{n}b_kF_k\left(\frac{1}{z^*}\right)$$
$$= f_0^+\left[\phi\left(z^*\right)\right]-\sum_{k=0}^{n}a_k\left[\phi\left(z^*\right)\right]^k\left(\phi'\left(z^*\right)\right)^{\frac{1}{p\left(z^*\right)-\varepsilon}}$$
$$+\frac{1}{2}\left(\phi'\left(z^*\right)\right)^{\frac{1}{p\left(z^*\right)-\varepsilon}}\left(f_0^+\left[\phi\left(z^*\right)\right]-\sum_{k=0}^{n}a_k\left[\phi\left(z^*\right)\right]^k\right)$$
$$+S_{\Gamma_1}\left[\left(\phi'\right)^{\frac{1}{p-\varepsilon}}\left(f_0^+\circ\phi-\sum_{k=0}^{n}a_k\phi^k\right)\right]\left(z^*\right)$$
$$-\frac{1}{2\pi i}\int_{\Gamma_2}\frac{f_1^+\left[\phi_1\left(\xi\right)\right]-\sum_{k=1}^{n}b_k\left[\phi_1\left(\xi\right)\right]^k}{\xi-z}d\xi \qquad (12)$$

a.e. on Γ_1.

Now using (12), Minkowski's inequality and the boundedness of S_{Γ_1} in $L^{p(\cdot),\theta}\left(\Gamma_1,\rho\right)$[43] we get

$$\left\|f-R_n\left(\cdot,f\right)\right\|_{L^{p),\theta}\left(\Gamma_1,\rho\right)}$$
$$\le c_5\left\|f_0^+\left(\omega\right)-\sum_{k=0}^{n}a_k\omega^k\right\|_{L^{p_0,\theta}\left(\mathbb{T},\rho_0\right)}+c_6\left\|f_1^+\left(\omega\right)-\sum_{k=0}^{n}b_k\omega^k\right\|_{L^{p_1,\theta}\left(\mathbb{T},\rho_1\right)} \qquad (13)$$

That is, the Faber-Laurent coefficients a_k and b_k of the function f are the Taylor coefficients of the functions f^+ and f_1^+, respectively. Then by (13), Lemma 1.1 and Theorem 1.2 we obtain

$$\left\|f-R_n\left(\cdot,f\right)\right\|_{L^{p(\cdot),\theta}\left(\Gamma_1,\rho\right)}\le c_7\left(p\right)\left\{\Omega^r_{p_0,\theta,\rho_0}\left(f_0,1/n\right)+\Omega^r_{p_1,\theta,\rho_1}\left(f_1,1/n\right)\right\}.$$

Let $z\in\operatorname{int}\Gamma_2$. Consideration of (3) and (9) gives us

$$\sum_{k=1}^{n} b_k F_{k,p}\left(\frac{1}{z}\right)$$
$$= \left(\phi_1'(z)\right)^{\frac{1}{p(z)-\varepsilon}}\left(\phi_1(z)\right)^{-\frac{2}{p(z)-\varepsilon}}\sum_{k=1}^{n} b_k\left[\phi_1(z)\right]^k$$
$$-\frac{1}{2\pi i}\int_{\Gamma_2}\frac{\left(\phi_1'(\zeta)\right)^{\frac{1}{p(\zeta)-\varepsilon}}\left(\phi_1(\zeta)\right)^{-\frac{2}{p(\zeta)-\varepsilon}}\sum_{k=1}^{n} b_k\left[\phi_1(\xi)\right]^k}{\xi-z}d\xi \qquad (14)$$
$$= \left(\phi_1'(z)\right)^{\frac{1}{p(z)-\varepsilon}}\left(\phi_1(z)\right)^{-\frac{2}{p(z)-\varepsilon}}\sum_{k=1}^{n} b_k\left[\phi_1(z)\right]^k$$
$$-\frac{1}{2\pi i}\int_{\Gamma_2}\frac{\left(\phi_1'(\zeta)\right)^{\frac{1}{p(\zeta)-\varepsilon}}\left(\phi_1(\zeta)\right)^{-\frac{2}{p(\zeta)-\varepsilon}}\left(\sum_{k=1}^{n} b_k\left[\phi_1(\xi)\right]^k - f_1^+\left[\phi_1(\xi)\right]\right)}{\xi-z}d\xi$$
$$-\frac{1}{2\pi i}\int_{\Gamma_2}\frac{f(\xi)}{\xi-z}d\xi - f_1^-\left[\phi_1(z)\right]\left(\phi_1'(z)\right)^{\frac{1}{p(z)-\varepsilon}}\left(\phi_1(z)\right)^{-\frac{2}{p(z)-\varepsilon}}.$$

For $z\in\operatorname{int}\Gamma_1$, using (1) and (9) we obtain

$$\sum_{k=1}^{n} a_k\Phi_k(z)$$
$$=\frac{1}{2\pi i}\int_{\Gamma_1}\frac{\left(\phi'(\zeta)\right)^{\frac{1}{p(\zeta)}}\sum_{k=1}^{n} a_k\left[\phi(\zeta)\right]^k}{\zeta-z}d\zeta \qquad (15)$$
$$=\frac{1}{2\pi i}\int_{\Gamma_1}\frac{\left(\phi'(\zeta)\right)^{\frac{1}{p(\zeta)}}\left(\sum_{k=1}^{n} a_k\left[\phi(\zeta)\right]^k - f_0^+\left[\phi(\zeta)\right]\right)}{\zeta-z}d\zeta$$
$$+\frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(\zeta)}{\zeta-z}d\zeta.$$

Now, by virtue of (14) and (15) for $z\in\operatorname{int}\Gamma_2$, we conclude that

$$\sum_{k=0}^{n} a_k \Phi_k(z) + \sum_{k=1}^{n} b_k F_k\left(\frac{1}{z}\right)$$
$$= \frac{1}{2\pi i}\int_{\Gamma_1} \frac{\left(\phi'(\zeta)\right)^{\frac{1}{p(\zeta)-\varepsilon}} \left(\sum_{k=0}^{n} a_k \left[\phi(\zeta)\right]^k - f_0^+\left[\phi(\zeta)\right]\right)}{\zeta - z} d\zeta$$
$$+ \left(\phi_1'(z)\right)^{\frac{1}{p(z)-\varepsilon}} \left(\phi_1(z)\right)^{-\frac{2}{p(z)-\varepsilon}} \sum_{k=1}^{n} b_k \left[\phi_1(z)\right]^k$$
$$- \frac{1}{2\pi i}\int_{\Gamma_2} \frac{\left(\phi_1'(\zeta)\right)^{\frac{1}{p(\zeta)-\varepsilon}} \left(\phi_1(\zeta)\right)^{-\frac{2}{p(\zeta)-\varepsilon}} \left[\sum_{k=1}^{n} b_k \left[\phi_1(\xi)\right]^k - f_1^+\left(\phi_1(\xi)\right)\right]}{\xi - z} d\xi$$
$$- f_1^-\left[\phi_1(z)\right]\left(\phi_1'(z)\right)^{\frac{1}{p(z)-\varepsilon}} \left(\phi_1(z)\right)^{-\frac{2}{p(z)-\varepsilon}}.$$

Taking limit as $z \to z^* \in \Gamma_2$ along all non-tangential paths outside Γ_2, we reach

$$f\left(z^*\right) - \sum_{k=0}^{n} a_k \Phi_{k,p}\left(z^*\right) - \sum_{k=1}^{n} b_k F_{k,p}\left(\frac{1}{z^*}\right)$$
$$= f_1^+\left[\phi_1\left(z^*\right)\right] - \frac{1}{2}\left(\phi_1'\left(z^*\right)\right)^{\frac{1}{p(z^*)-\varepsilon}} \left(\phi_1\left(z^*\right)\right)^{-\frac{2}{p(z^*)-\varepsilon}} \left[\sum_{k=1}^{n} b_k \left[\phi_1\left(z^*\right)\right]^k - f_1^+\left[\phi_1\left(z^*\right)\right]\right]$$
$$- S_{\Gamma_2}\left[\left(\phi_1'\right)^{\frac{1}{p}} \left(\phi_1\right)^{-\frac{2}{p-\varepsilon}} \left(\sum_{k=1}^{n} b_k \phi_1^k\right) - \left(f_1^+ \mathrm{o} \phi_1\right)\right]\left(z^*\right)$$
$$- \frac{1}{2\pi i}\int_{\Gamma_1} \frac{\phi'(\zeta)^{\frac{1}{p(\zeta)-\varepsilon}} \left(\sum_{k=0}^{n} a_k \left[\phi(\zeta)\right]^k - f_0^+\left[\phi(\zeta)\right]\right)}{\zeta - z^*} d\zeta \qquad (16)$$

a.e. on Γ_2.

Using (16), Minkowski's inequality and the boundedness of S_{Γ_2} in $L^{p(.),\theta}\left(\Gamma_2, \rho\right)$
[40] we get

$$\left\| f - R_n(\cdot, f)\right\|_{L^{p(.),\theta}(\Gamma_2,\rho)}$$
$$\le c_8 \left\| f_1^+(\omega) - \sum_{k=1}^{n} b_k \omega^k \right\|_{L^{p_1,\theta}(\mathbb{T},\rho_1)} + c_9 \left\| f_0^+\left(\omega - \sum_{k=0}^{n} a_k \omega^k\right)\right\|_{L^{p_0,\theta}(\mathbb{T},\rho_0)} \qquad (17)$$

Use of (17) and Lemma 1.1 and Theorem 1.2 leads to

$$\left\| f - R_n(\cdot, f) \right\|_{L^{p(\cdot),\theta}(\Gamma_2,\rho)} \le c_{10} \left\{ \Omega^r_{p_1,\theta,\rho_1}(f_1, 1/n) + \Omega^r_{p_0,\theta,\rho_0}(f_0, 1/n) \right\}.$$

The proof is complete.

3. Conclusion

Variable exponential Lebesgue spaces $L^{p(\cdot)}$, known as generalizations of Lebesgue spaces, appeared in literature for the first time in 1931 with an article written by Orlicz [46]. Note that the generalized Lebesgue spaces with variable exponents are used in the theory of elasticity, in mechanics, especially in fluid dynamics for the modelling of electrorheological fluids, in the theory of diferential operators, and in variational calculus (see, for example, [10], [11], [12], [48] and [50]). We investigate the approximation properties of the functions by Faber-Laurent rational functions in the $\rho-$ power weighted grand variable exponent Smirnov classes defined in the doubly connected domain of the complex plane.

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